Math 246C Lecture 27 Notes

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May 30, 2019

1 L^2-Estimates for the $\overline{\partial}$-Operator: The Density Lemma

1.1 The density lemma

In solving our $\overline{\partial}$ problem, we have

$$L^2(\Omega, e^{-\varphi_1}) \xrightarrow{T} L^2_{(0,1)}(\Omega, e^{-\varphi_2}) \xrightarrow{S} L^2_{(0,2)}(\Omega, e^{-\varphi_3}).$$

We want to show that

$$\|f\|_{\varphi_2} \leq C(\|T^*f\|^2_{\varphi_1} + \|Sf\|^2_{\varphi_3}), \qquad \forall f \in D(T^*) \cap D(S).$$

We had the following lemma:

Lemma 1.1 (Density lemma)**.** *Let (η_ν) be a sequence in $C_0^\infty(\Omega)$ such that $0 \leq \eta_\nu \leq 1$ and such that for any compact $K \subseteq \Omega$, $\eta_\nu = 1$ on K for all large ν. Assume that*

$$e^{-\varphi_{j+1}} |\overline{\partial}\eta_\nu|^2 \leq C e^{-\varphi_j}, \qquad \forall \nu, j = 1, 2.$$

Then $C^\infty_{0,(0,1)}(\Omega)$ is dense in $D(T^*) \cap D(S)$ with respect to the graph norm.

Proof. Step 1: Suppose $f \in D(T^*) \cap D(S)$ has compact support. Approximate by $f * \psi_\varepsilon$, where $\psi_\varepsilon(z) = \varepsilon^{-2n}\psi(z/\varepsilon)$ and $\psi \in C_0^\infty$.

Step 2: Let $f \in D(T^*) \cap D(S)$. We claim that $\eta_j f \in D(T^*) \cap D(S)$. To show that $\eta_j f \in D(S)$,

$$\overline{\partial}(\eta_j f) = \underbrace{\eta_j \underbrace{\overline{\partial} f}_{\in L^2_{\varphi_3}}}_{\in L^2_{\varphi_3}} + \underbrace{\overline{\partial}\eta_j \wedge f}_{\in L^2_{\varphi_3}}.$$

To show that $\eta_j f \in D(T^*)$, consider for $u \in D(T)$,

$$\langle Tu, \eta_j f\rangle_{\varphi_2} = \langle \eta_j Tu, f\rangle_{\varphi_2}$$

Observe that $\eta_j Tu = \eta_j \overline{\partial} u = \overline{\partial}(\eta_j u) - u\overline{\partial}\eta_j$, where $\eta_j u \in D(T)$.

$$
\begin{aligned}
&= \langle T(\eta_j u), f\rangle_{\varphi_2} - \int u \left\langle \overline{\partial}\eta, f\right\rangle e^{-\varphi_2} \\
&= \langle u, \eta_j T^* f\rangle_{\varphi_1} - \left\langle u, e^{\varphi_1 - \varphi_2} \left\langle \overline{\partial}\eta, f\right\rangle\right\rangle_{\varphi_1}.
\end{aligned}
$$

So

$$
T^*(\eta_j f) = \eta_j T^* f - e^{-\varphi_1 - \varphi_2} \left\langle \overline{\partial}\eta, f\right\rangle.
$$

We now check that $\eta_j f \to f$ in the graph norm.

1. $\eta_j f \to f$ in $L^2_{\varphi_2}$: This follows by the dominated convergence theorem.

2. $S(\eta_j f) \to Sf$ in L_{φ_3}: We have

$$
S(\eta_j f) = \overline{\partial}(\eta_j f) = \underbrace{\eta_j \underbrace{Sf}_{\in L^2_{\varphi_3}}}_{\to Sf \text{ in } L^2_{\varphi_3}} + \underbrace{\overline{\partial}\eta_j \wedge f}_{\to 0 \text{ in } L^2_{\varphi_3}}
$$

   So we get that

$$
\int \underbrace{|\overline{\partial}\eta_j|^2 e^{-\varphi_3}}_{\le e^{-\varphi_2}} |f|^2 \to 0
$$

   by the dominated convergence theorem.

3. $T^*(\eta_j f) \to T^* f$ in $L^2_{\varphi_1}$ is similar. □

1.2 Applying the lemma

Now let $\psi \in C^\infty(\Omega)$ be given by the locally finite sum

$$
e^\psi = 1 + \sum_{\nu=1}^{\infty} |\overline{\partial}\eta_\nu|^2.
$$

Let $\varphi_j = \varphi + (j-3)\psi$ for $j = 1, 2, 3$ (φ is to be chosen). With this choice of weights, we can satisfy the hypotheses of the density lemma.

We will now study our estimate

$$
\|f\|^2_{\varphi_2} \le C(\|T^* f\|^{@}_{\varphi_1} + \|Sf\|^2_{\varphi_2}), \qquad f \in C_0^\infty.
$$

Recall the formula for T^*:

$$
T^* f = -e^{\varphi_1} \sum_{j=1}^{\infty} \partial_{z_j}(f_j e^{-\varphi_2}) = -e^{\varphi - 2\psi} \sum_{j=1}^{\infty} \partial_{z_j}(f_j e^{\psi - \varphi}).
$$

Then

$$e^{\psi}T^*f = -\sum \delta_j f_j - \sum f_j \partial_{z_j}\psi, \qquad \delta_j := \partial_{z_j} - \partial_{z_j}\varphi.$$

Here, $-\delta_j$ is the adjoint of $\partial_{\overline{z}_j}$ in L^2_φ.

Consider

$$\|T^*f\|^2_{\varphi_1} = \int |T^*f|^2 e^{-\varphi+2\psi} = \|e^{\psi}T^*f\|_{\varphi}.$$

Then, using Cauchy-Schwarz or the triangle inequality,

$$\begin{aligned}\left\|\sum \delta_j f_j\right\|^2_{\varphi} &= \|e^{\psi}T^*f + \langle f, \partial\psi\rangle\|^2_{\varphi} \\ &\leq 2\|T^*f\|^2_{\varphi_1} + 2\int |\langle t, \partial\psi\rangle|^2 e^{-\varphi}.\end{aligned}$$

Compute $\|Sf\|^2_{\varphi_3}$:

$$Sf = \overline{\partial} f = \sum_{j<k}\left(\frac{\partial d_k}{\partial \overline{z}_j} - \frac{\partial f_j}{\partial \overline{z}_k}\right) d\overline{z}_j \wedge d\overline{z}_k.$$

So

$$\begin{aligned}\|Sf\|^2_{\varphi_3} &= \sum_{j<k}\int \left|\frac{\partial f_k}{\partial \overline{z}_j} - \frac{\partial f_j}{\partial \overline{z}_k}\right|^2 e^{-\varphi} \\ &= \frac{1}{2}\sum_{j,k}\int \left|\frac{\partial f_k}{\partial \overline{z}_j} - \frac{\partial f_j}{\partial \overline{z}_k}\right|^2 e^{-\varphi} \\ &= \int \sum_{j,k}\left|\frac{\partial f_k}{\partial \overline{z}_j}\right|^2 e^{-\varphi} - \left(\sum_{j,k}\frac{\partial f_j}{\partial \overline{z}_k}\overline{\frac{\partial f_k}{\partial \overline{z}_j}}\right) e^{-\varphi}\end{aligned}$$

Add $\|Sf\|^2_{\varphi_3}$ to both sides of the inequality. We get the following estimate:

$$\left\|\sum \delta_j f_j\right\|^2_{\varphi} - \sum_{j,k}\left\langle \partial_{\overline{z}_k} f_j, \partial_{\overline{z}_j} f_k\right\rangle_{\varphi} \leq 2\|T^*f\|^2_{\varphi_1} + 2\int |\langle f, \partial\psi\rangle|^2 e^{-\varphi} + \|Sf\|^2_{\varphi_3}.$$

The main point of the argument is that

$$\begin{aligned}\langle \delta_j f_j, \delta_k, f_k\rangle_{\varphi} - \left\langle \partial_{\overline{z}_k} f_j, \partial_{\overline{z}_j} f_k\right\rangle_{\varphi} &= -\langle \partial_{\overline{z}_k}\delta_j f_j, f_k\rangle_{\varphi} + \left\langle \delta_{z_j}\partial_{\overline{z}_k} f_j, f_k\right\rangle_{\varphi} \\ &= \left\langle [\delta_{z_j}, \partial_{\overline{z}_k}] f_j, f_k\right\rangle_{\varphi}.\end{aligned}$$

The commutator equals

$$[\partial_{z_j} - \partial_{z_j}\varphi, \partial_{\overline{z}_k}] = \frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k}.$$

So the lower bound becomes

$$\int \sum_{j,k} \frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k} f_j f_k e^{-\varphi},$$

where $\frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k}$ is the Levi form of $\varphi(f)$. Now we can choose φ to be plurisubharmonic.

We will conclude our discussion next time.